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EFFECTS OF INTERLAMINAR SHEAR ON THE BENDING AND BUCKLING OF LAMINATED BEAMS

By

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J. Mayers

March 1970

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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STANFORD UNIVERSITY
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This program was carried out under Contract DAAJ02-68-C-0035 with Stanford University.

The research was directed toward the development of techniques for predicting the effects of interlaminar shear on the bending and buckling of laminated beams. Two specific problems are studied: bending under uniform load and buckling of simply supported beams.

The report has been reviewed by the U. S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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March 1970

EFFECTS OF INTERLAMINAR SHEAR ON THE BENDING AND BUCKLING OF LAMINATED BEAMS

Final Report

By

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SUMMARY

The governing equations for laminated beams are developed by variational methods. The beams are considered to consist of n laminae and $n-1$ bond layers, where n can be any reasonable number greater than one. The bond thicknesses are assumed to be small in comparison with those of the laminae. The effects of shear strains and direct strains normal to the bending axis in the bond layers are included. Two specific problems are solved: bending under uniform load and buckling of simply supported beams. Curves are presented which show the effect of bond flexibility.

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FOREWORD

The work reported herein constitutes a portion of a continuing effort being undertaken at Stanford University for the U. S. Army Aviation Materiel Laboratories under Contract DAAJ02-68-C-0035 (Task 1F162204A17002) to establish accurate theoretical prediction capability for the static and dynamic behavior of aircraft structural components utilizing both conventional and unconventional materials.

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LIST OF SYMBOLS

$[A],[B]$	matrices
$[A]_j,[B]_j$	matrices corresponding to jth term of solution series
a_i,d_i	coefficients of displacement functions for ith lamina, in.
a_{ij},d_{ij}	coefficients of jth term of displacement functions for ith lamina, in.
$(a_i),(d_i)$	vectors of lamina displacement coefficients
$(a_i)_j,(d_i)_j$	vectors of lamina displacement coefficients corresponding to jth term of displacement functions
b	width of beam, in.
(C)	vector
$(C)_j$	vector corresponding to jth term of solution series
c	thickness of lamina, in.
E	extensional modulus for lamina material, lb/in.^{2}
E_B	extensional modulus for bond material in direction normal to bending axis, lb/in.^{2}
G_B	shear modulus for bond material, lb/in.^{2}
I	moment of inertia
I_o	moment of inertia of single lamina, in.^{4}
j	term of displacement function
k	physical constant as defined in text
L	length of laminated beam, in.
n	number of laminae in laminated beam
P	axial load, lb
p	uniform load density, lb/in.^{2}
Q	first moment of an area, in.^{3}

t	thickness of bond layer, in.
U	strain energy functional, lb-in.
u_i	axial displacement function for ith lamina, in.
V	shear force, lb
V_p	potential functional of uniformly distributed load, lb-in.
V_P	potential functional of axial load, lb-in.
w_i	transverse displacement function for ith lamina, in.
x	axial coordinate of laminated beam, in.
z	coordinate normal to middle surface of a lamina, in.
α	total angle of distortion of bond layer, radians
α_1	angle of distortion of bond layer due to bending of adjacent lamina, radians
γ_B	shear strain in bond layer, in./in.
Δ	maximum beam deflection, in.
δ	variational operator
δ_1	displacement of upper edge of bond layer, in.
δ_2	displacement of lower edge of bond layer, in.
δ_{i1}, δ_{in}	Kronecker symbol
ϵ_2	extensional strain in bond layer normal to bending axis, in./in.
σ_{max}	maximum extensional stress in bond layer, lb/in.^{2}
τ	shear stress, lb/in.^{2}

<u>SUBSCRIPTS</u>

B	bond layer
n	nth lamina
i	ith lamina
j	jth term of displacement function

INTRODUCTION

The bending of multilayer sandwich plates was investigated by Liaw and Little in Reference 1. In that paper the authors assume the facings to carry only axial loads and inplane shear (no bending), whereas the cores are assumed to carry only transverse shear stress. Stress resultants are then formulated which reflect the contributions of the cores and the facings subject to the above assumptions. Minimization of the total complementary energy, in a procedure similar to that carried out by Reissner[2] in presenting a theory for the bending of nonhomogeneous elastic plates, leads to the governing equations for a multilayer plate. Since the multilayer effect is contained in the stress resultants, the final equations for the stress resultants are the same as those for a single sandwich plate except for the definition of the physical constants.

Inherent in the Liaw and Little analysis is the assumption of the same shear angle for all core layers. The validity of this assumption was investigated by Kao and Ross[3]; they considered a beam composed of three facings and two core layers. By allowing the structure to have four degrees of freedom, an inplane displacement function for each face and an out-of-plane displacement function for the structure as a whole, the two core layers are then capable of having different shear angles. Minimization of the total potential energy with respect to each degree of freedom leads to the generation of the governing equations for the system.

In both References 1 and 3, the core depths must be considered large in comparison with the thickness of the facings. Although, in Reference 3, the bending capability of the facings is included, the strain-displacement relations for the core shears are not valid when the facings are thick since they do not include the effect of rotation of the facings. Reference 1 provides no way of accurately estimating the significant bond stresses, since the theory deals only with average effects across the overall plate depth. From the analysis in Reference 3, an estimate of the bond shear stress is possible; however, estimates of the peel strength of the bonds are not possible since the peel strength depends on the direct stress in the bond layer in the direction normal to the bending axis.

Also, in Reference 3, the theory and its application are limited to a beam consisting of three laminae.

This report deals with laminated beams in which the bond layers are small in comparison with the thickness of the laminae. By utilization of variational techniques to establish the governing equations and associated boundary conditions, the effect of the bond flexibility on the buckling and bending of laminated beams is studied. The analysis is performed for a beam consisting of n laminae and $n-1$ bond layers, where n can be any reasonable number greater than one. From the results of the bending analysis, the shear stresses and direct stresses normal to the bending axis in the bond layers can be calculated. Such information is of prime importance in the design of laminated beams.

The laminated beam is considered to have $2n$ degrees of freedom, an inplane and an out-of-plane displacement function corresponding to each lamina (see Figure 1). The total potential energy functional is then formulated in terms of these $2n$ displacement functions. Coupling of the individual laminae is achieved by describing the strain energy of the bond layers in terms of the displacement functions of the laminae they connect.

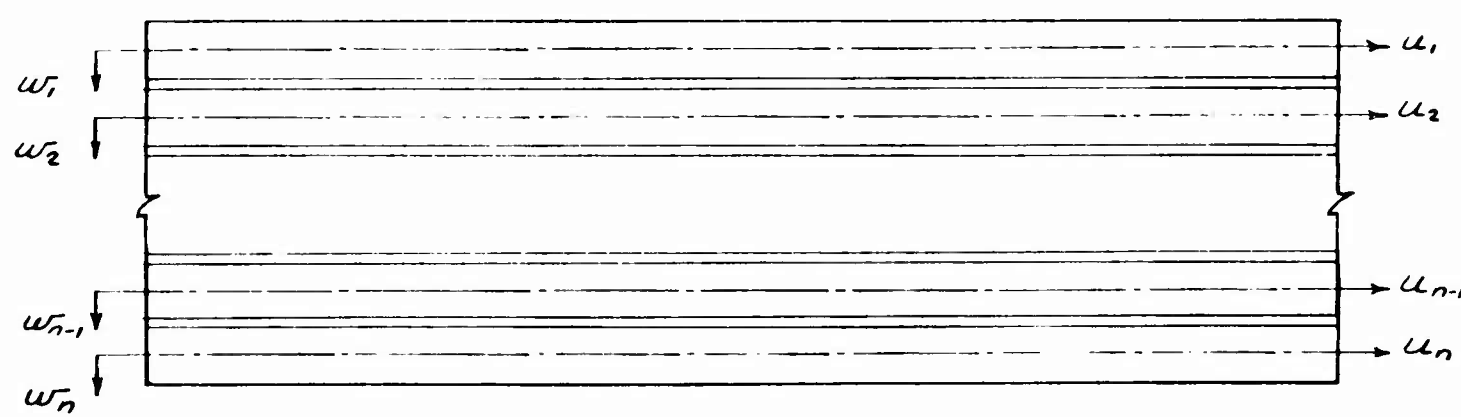

Figure 1. n-Ply Laminated Beam Described by 2n Displacement Functions.

BASIC THEORY

STATEMENT OF PROBLEM AND BASIC ASSUMPTIONS

The problem considered herein is that of analytically formulating the behavior of an n-ply laminated beam (see Figure 1) with nonrigid bond layers, and applying this formulation to the problem of bending of laminated beams under a uniformly distributed load and to the problem of buckling of laminated beams under axial loading.

The development is predicated on the following assumptions:

1. For the individual laminae, all of the assumptions made in the engineering theory of beams are assumed to apply.
2. The laminae all have the same elastic and geometric properties.
3. The bond layers all have the same elastic and geometric properties.
4. The end condition for the structure as a whole is that of simple support.
5. The individual laminae at the ends of the beam are not restrained in any way from axial movement.
6. The bending stresses and axial stresses parallel to the bending axis in the bonds are negligible.
7. The stress through the thickness of a bond layer is constant.

STRAIN-DISPLACEMENT RELATIONS FOR A BOND LAYER

As derived in Appendix I, the shear strain in a bond layer (see Figure 2) is given in terms of the axial and lateral displacements of the i and $i+1$ laminae as

$$\gamma_B = \frac{u_i - u_{i+1}}{t} - \frac{1}{2}\left(1 + \frac{c}{t}\right)\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right) \tag{1}$$

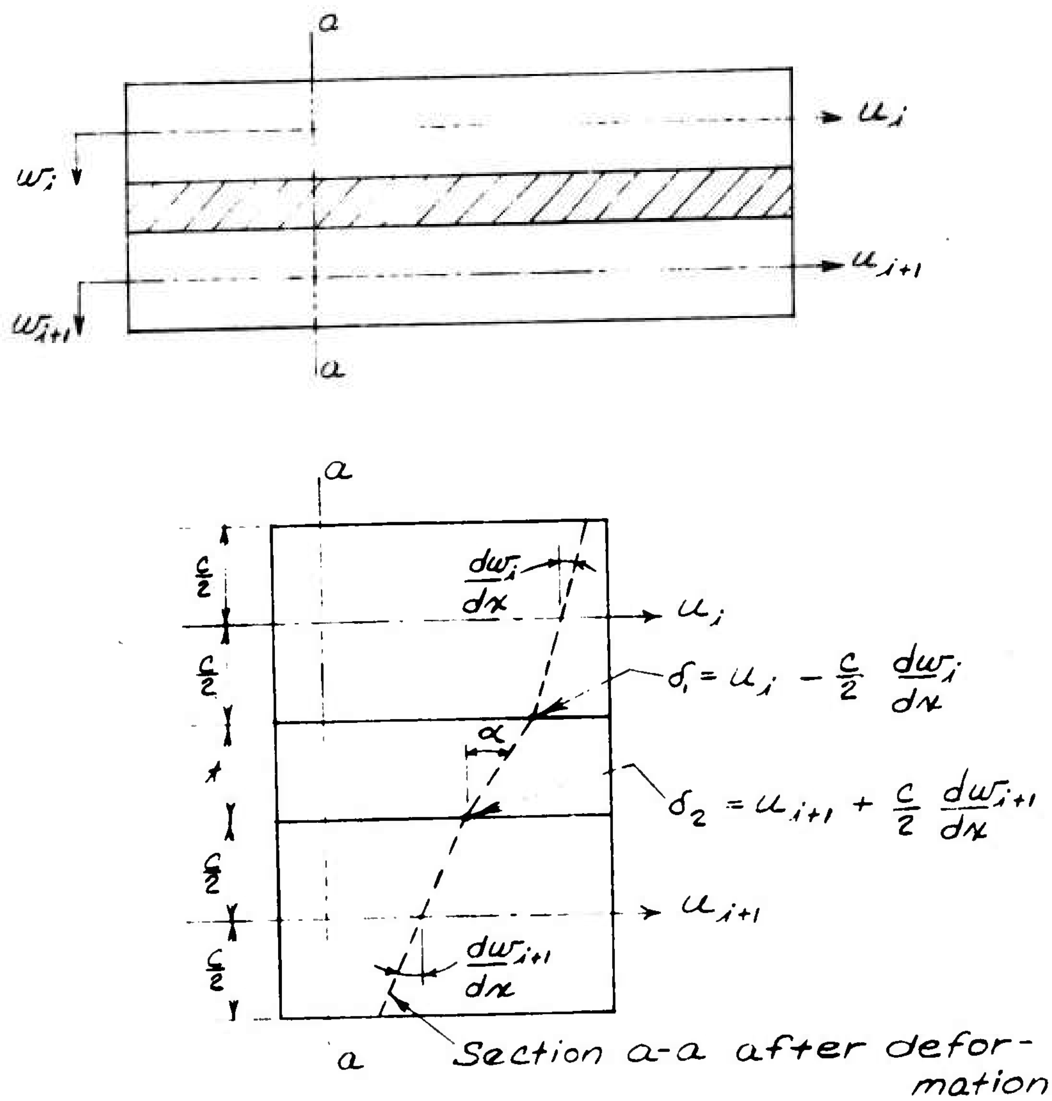

Figure 2. Geometry of Assumed Transverse Shear Deformation in Bond Layers of Laminated Beams.

STRAIN-DISPLACEMENT RELATIONS FOR A LAMINA

In accordance with elementary beam theory, the axial strain in the ith lamina is

$$\epsilon_x = \frac{du_i}{dx} - z\,\frac{d^2 w_i}{dx^2} \tag{2}$$

STRAIN ENERGY IN A LAMINATED BEAM

The strain energy stored in a laminated beam, with flexibility effects included, is

$$U = U_1 + U_2 + U_3 \tag{3}$$

where U_1 is the strain energy due to transverse shearing deformations of the bond layers, U_2 is the average strain energy in the bond layers due to the bonding agent compressibility in the thickness direction, and U_3 is the strain energy associated with stretching and bending of the laminae. The quantities U_1 and U_2 are derived in Appendix I. Thus.

$$U = \sum_{i=1}^{n-1}\left\{\frac{G_B b}{2t}\int_0^L\left[u_i - u_{i+1} - \frac{c+t}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right)\right]^2 dx\right\}$$

$$+ \sum_{i=1}^{n-1}\left\{\frac{E_B b}{2t}\int_0^L [w_{i+1} - w_i]^2 dx\right\}$$

$$+ \sum_{i=1}^{n}\left\{\frac{Ebc}{2}\int_0^L\left(\frac{du_i}{dx}\right)^2 dx + \frac{EI_o}{2}\int_0^L\left(\frac{d^2 w_i}{dx^2}\right)^2 dx\right\} \tag{4}$$

VARIATIONAL PRINCIPLE, GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Bending Under Distributed Loading

For the bending problem (see Figure 3), the total potential energy stored in the system is

$$U + V_p = \sum_{i=1}^{n-1}\left\{\frac{G_B b}{2t}\int_0^L\left[u_i - u_{i+1} - \frac{c+t}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right)\right]^2 dx\right\}$$

$$+ \sum_{i=1}^{n-1}\left\{\frac{E_B b}{2t}\int_0^L [w_{i+1} - w_i]^2 dx\right\}$$

$$+ \sum_{i=1}^{n}\left\{\frac{Ebc}{2}\int_0^L\left(\frac{du_i}{dx}\right)^2 dx + \frac{EI_o}{2}\left(\frac{dw_i}{dx}\right)^2 dx\right\} - \int_0^L pbw_1\, dx \tag{5}$$

where V_p is the potential of the distributed load acting over the surface of the $i=1$ laminae.

As developed in Appendix,II the extremes of $U + V_p$ with respect to the 2n degrees of freedom u_i and w_i $(i = 1,2,\ldots,n)$ are the 2n Euler equations

$$EI_o \frac{d^4 w_i}{dx^4} - (1-\delta_{i1}-\delta_{in})\frac{E_B b}{t}(-w_{i-1} + 2w_i - w_{i+1}) + (\delta_{i1})\frac{E_B b}{t}(w_1 - w_2)$$

$$+(\delta_{in})\frac{E_B b}{t}(w_n - w_{n-1}) - (\delta_{in})\frac{G_B b(c+t)}{2t}\left[\frac{du_n}{dx} - \frac{du_{n-1}}{dx}\right.$$

$$\left. + \frac{c+t}{2}\left(\frac{d^2 w_n}{dx^2} + \frac{d^2 w_{n-1}}{dx^2}\right)\right] - (\delta_{i1})\frac{G_B b(c+t)}{2t}\left[\frac{du_2}{dx} - \frac{du_1}{dx}\right.$$

$$\left. + \frac{c+t}{t}\left(\frac{d^2 w_2}{dx^2} + \frac{d^2 w_1}{dx^2}\right)\right] - (1-\delta_{i1}-\delta_{in})\frac{G_B b(c+t)}{2t}\left[\frac{du_{i+1}}{dx} - \frac{du_{i-1}}{dx}\right.$$

$$\left. + \frac{c+t}{t}\left(\frac{d^2 w_{i-1}}{dx^2} + 2\frac{d^2 w_i}{dx^2} + \frac{d^2 w_{i+1}}{dx^2}\right)\right] - (\delta_{i1})p = 0$$

$$i = 1,2,\ldots,n$$

(continued)

$$-Ebc\,\frac{d^2u_i}{dx^2} - (\delta_{i1})\,\frac{G_Bb}{t}\left[u_2-u_1+\frac{c+t}{2}\left(\frac{dw_2}{dx}+\frac{dw_1}{dx}\right)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\,\frac{G_Bb}{t}\left[-u_{i-1}+2u_i-u_{i+1}+\frac{c+t}{2}\left(\frac{dw_{i-1}}{dx}-\frac{dw_{i+1}}{dx}\right)\right]$$

$$+(\delta_{in})\,\frac{G_Bb}{t}\left[u_n-u_{n-1}+\frac{c+t}{2}\left(\frac{dw_n}{dx}+\frac{dw_{n-1}}{dx}\right)\right]=0$$

$$i = 1,2,\ldots,n \qquad (6)$$

The associated boundary conditions obtained from the variational procedure carried out in Appendix II are the 3n relations

$$EI_o\,\frac{d^2w_i}{dx^2}\Bigg|_0^L = 0 \qquad \text{or} \qquad \frac{d}{dx}\,\delta w_i\Bigg|_0^L = 0 \qquad i = 1,2,\ldots,n$$

$$\Bigg\{-EI_o\,\frac{d^3w_i}{dx^3} + (\delta_{i1})\,\frac{G_Bb(c+t)}{2t}\left[u_2-u_1+\frac{c+t}{2}\left(\frac{dw_1}{dx}+\frac{dw_2}{dx}\right)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\,\frac{G_Bb(c+t)}{2t}\left[u_{i+1}-u_{i-1}+\frac{c+t}{2}\left(\frac{dw_{i+1}}{dx}+2\,\frac{dw_i}{dx}+\frac{dw_{i-1}}{dx}\right)\right]$$

$$+\delta_{in}\,\frac{G_Bb(c+t)}{2t}\left[u_n-u_{n-1}+\frac{c+t}{2}\left(\frac{dw_n}{dx}+\frac{dw_{n-1}}{dx}\right)\right]\Bigg\}\Bigg|_0^L = 0$$

or

$$\delta w_i\Bigg|_0^L = 0 \qquad i = 1,2,\ldots,n$$

$$Ebc\,\frac{du_i}{dx}\Bigg|_0^L = 0 \qquad \text{or} \qquad \delta u_i\Bigg|_0^L = 0 \qquad i = 1,2,\ldots,n \qquad (7)$$

Buckling Under Axial Loading

For the beam buckling problem (see Figure 4), the total potential energy stored in the system is

$$U + V_P = \sum_{i=1}^{n-1} \left\{ \frac{G_B b}{2t} \int_0^L \left[u_i - u_{i+1} - \frac{c+t}{2} \left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx} \right) \right]^2 dx \right\}$$

$$+ \sum_{i=1}^{n-1} \left\{ \frac{E_B b}{2t} \int_0^L [w_{i+1} - w_i]^2 dx \right\}$$

$$+ \sum_{i=1}^{n} \left\{ \frac{Ebc}{2t} \int_0^L \left(\frac{du_i}{dx} \right)^2 dx + \frac{EI_o}{2} \int_0^L \left(\frac{d^2 w_i}{dx^2} \right)^2 dx \right\}$$

$$- \sum_{i=1}^{n} \left\{ \frac{P}{n} \int_0^L \frac{1}{2} \left(\frac{dw_i}{dx} \right)^2 dx \right\} \tag{8}$$

where V_P is the potential of the axial load applied over the end surfaces of the beam laminae. The particular relation for V_P is presented in Appendix III.

As developed in Appendix III, the extreme of $U + V_P$ with respect to the $2n$ degrees of freedom u_i and w_i $(i = 1,2,\ldots n)$ are the $2n$ Euler equations

$$EI_o \frac{d^4 w_i}{dx^4} - (1-\delta_{i1}-\delta_{in}) \frac{E_B b}{t} (-w_{i-1} + 2w_i - w_{i+1}) + (\delta_{i1}) \frac{E_B b}{t} (w_1 - w_2)$$

$$+ (\delta_{in}) \frac{E_B b}{t} (w_n - w_{n-1}) - (\delta_{in}) \frac{G_B b(c+t)}{2t} \left[\frac{du_n}{dx} - \frac{du_{n-1}}{dx} \right.$$

$$\left. + \frac{c+t}{2} \left(\frac{d^2 w_n}{dx^2} + \frac{d^2 w_{n-1}}{dx^2} \right) \right] - (\delta_{i1}) \frac{G_B b(c+t)}{2t} \left[\frac{du_2}{dx} - \frac{du_1}{dx} \right.$$

(continued)

$$+ \frac{c+t}{2}\left(\frac{d^2w_2}{dx^2} + \frac{d^2w_1}{dx^2}\right)\Bigg] - (1-\delta_{i1}-\delta_{in})\ \frac{G_Bb(c+t)}{2t}\left[\frac{du_{i+1}}{dx} - \frac{du_{i-1}}{dx}\right.$$

$$\left. + \frac{c+t}{2}\left(\frac{d^2w_{i-1}}{dx^2} + 2\,\frac{d^2w_i}{dx^2} + \frac{d^2w_{i+1}}{dx^2}\right)\right] + \frac{P}{n}\,\frac{d^2w_i}{dx^2} = 0, \quad i=1,2,\ldots,n$$

$$-Ebc\,\frac{d^2u_i}{dx^2} - (\delta_{i1})\ \frac{G_Bb}{t}\left[u_2-u_1 + \frac{c+t}{2}\left(\frac{dw_2}{dx} + \frac{dw_1}{dx}\right)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\ \frac{G_Bb}{t}\left[-u_{i-1} + 2u_i - u_{i+1} + \frac{c+t}{2}\left(\frac{dw_{i-1}}{dx} - \frac{dw_{i+1}}{dx}\right)\right]$$

$$+(\delta_{in})\ \frac{G_Bb}{t}\left[u_n-u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right] = 0 \qquad i=1,2,\ldots,n$$

(9)

The associated boundary conditions obtained from the variational procedure carried out in Appendix III are the 3n relations

$$EI_o\,\frac{d^2w_i}{dx^2}\Bigg|_0^L = 0 \qquad \text{or} \qquad \frac{d}{dx}\,\delta w_i\Bigg|_0^L = 0 \qquad i=1,2,\ldots,n$$

$$\Bigg\{-EI_o\,\frac{d^3w_i}{dx^3} + (\delta_{i1})\ \frac{G_Bb(c+t)}{2t}\left[u_2-u_1 + \frac{c+t}{2}\left(\frac{dw_1}{dx} + \frac{dw_2}{dx}\right)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\ \frac{G_Bb(c+t)}{2t}\left[u_{i+1}-u_{i-1} + \frac{c+t}{2}\left(\frac{dw_{i+1}}{dx} + 2\,\frac{dw_i}{dx} + \frac{dw_{i-1}}{dx}\right)\right]$$

$$+(\delta_{in})\ \frac{G_Bb(c+t)}{2t}\left[u_n-u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right] - \frac{P}{n}\,\frac{dw_i}{dx}\Bigg\}_0^L = 0$$

$$\text{or} \qquad \delta w_i\Big|_0^L = 0 \qquad i=1,2,\ldots,n$$

(continued)

$$\left. Ebc \frac{du_i}{dx} \right|_0^L = 0 \qquad \text{or} \qquad \left. \delta u_i \right|_0^L = 0 \qquad\qquad i=1,2,\ldots,n \quad (10)$$

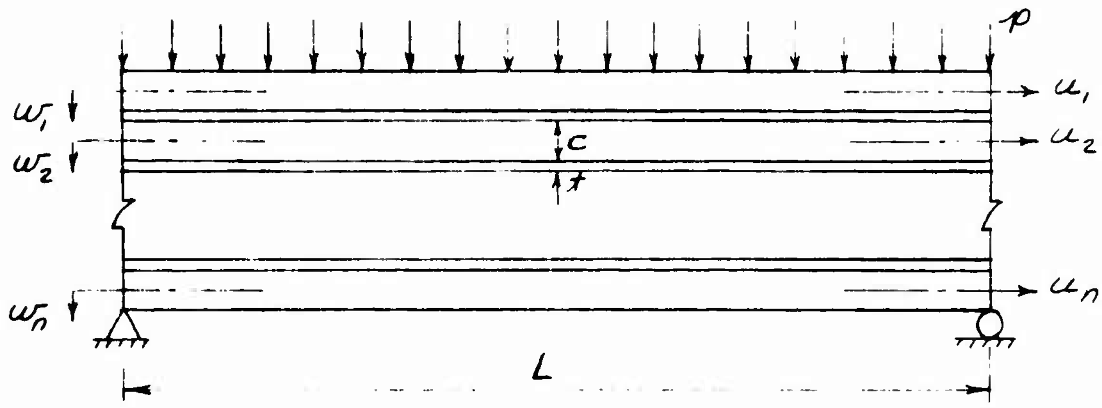

Figure 3. Laminated Beam Under Uniformly Distributed Loading.

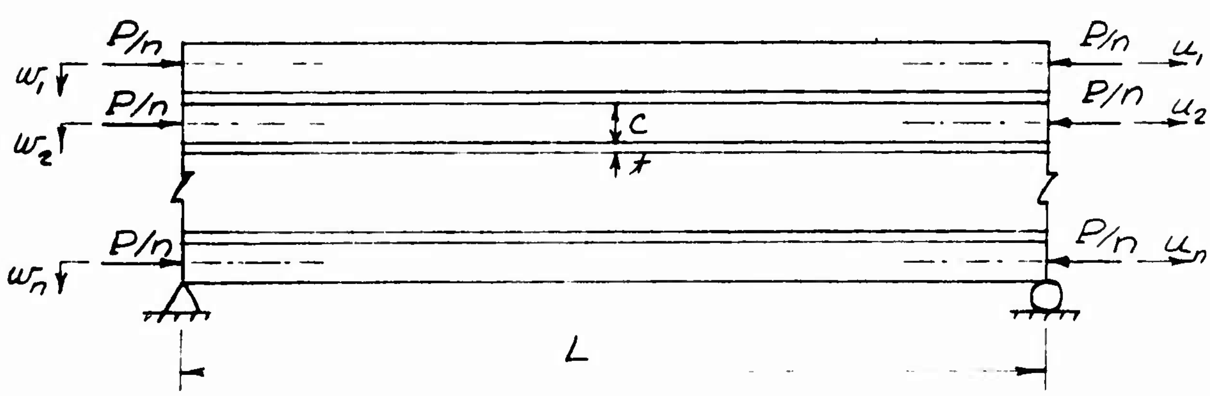

Figure 4. Laminated Beam Under Axial Loading.

METHOD OF SOLUTION

For the two problems being investigated, the bending of a laminated beam under lateral loading and the buckling of a simply supported laminated beam under axial loading, the governing equations and boundary conditions are prescribed in equations (6) and (7) and (9) and (10) respectively.

BENDING UNDER DISTRIBUTED LOADING

In the bending problem, the complexity of the equilibrium equations (6) precludes straightforward integration and solution consistent with the boundary conditions of simple support. Thus w_i and u_i are assumed as

$$w_i = a_{ij} \sin \frac{j\pi x}{L}$$

$$i = 1,2,\ldots,n$$

$$j = 1,2,\ldots,\infty \tag{11}$$

$$u_i = d_{ij} \cos \frac{j\pi x}{L}$$

$$i = 1,2,\ldots,n$$

$$j = 1,2,\ldots,\infty \tag{12}$$

and used in conjunction with the total potential energy, equation (5), to effect a Rayliegh-Ritz approximate solution. Both u_i and w_i have been selected to satisfy the geometric boundary conditions for simple support. As shown in Appendix IV, the application of the minimum total potential energy principle leads to the equilibrium relations

$$\left\{\frac{EI_o}{2}\left(\frac{j\pi}{L^3}\right)^4 + \left[\frac{E_B bL}{2t} + \frac{G_B b}{t}\frac{[j\pi(c+t)]^2}{8L}\right](2-\delta_{i1}-\delta_{in})\right\} a_{ij}$$

$$+\left\{-\frac{E_B bL}{2t} + \frac{G_B b}{t}\frac{[j\pi(c+t)]^2}{8L}\right\}(1-\delta_{i1})a_{i-1\,j}$$

(continued)

$$+\left\{-\frac{E_B bL}{2t}+\frac{G_B b}{t}\frac{[j\pi(c+t)]^2}{8L}\right\}(1-\delta_{in})a_{i+1\ j}$$

$$+\left\{-\frac{G_B b}{t}\cdot\frac{j\pi(c+t)}{4}\right\}(1-\delta_{i1})d_{i-1\ j}$$

$$+\left\{\frac{G_B b}{t}\cdot\frac{j\pi(c+t)}{4}\right\}(1-\delta_{in})d_{i+1\ j}$$

$$+\left\{\frac{G_B b}{t}\cdot\frac{j\pi(c+t)}{4}\right\}(\delta_{in}-\delta_{i1})d_{ij}-\frac{2Lp}{j\pi}(\delta_{i1})=0$$

$$i=1,2,\ldots,n$$

$$j=1,2,\ldots,\infty \tag{13}$$

$$\left[\frac{j\pi\ G_B b(c+t)}{4t}\right](1-\delta_{i1})a_{i-1\ j}+\left[\frac{j\pi\ G_B b(c+t)}{4t}\right](\delta_{in}-\delta_{i1})a_{ij}$$

$$+\left[\frac{j\pi\ G_B b(c+t)}{4t}\right](1-\delta_{in})a_{i+1\ j}+\left[-\frac{G_B bL}{2t}\right](1-\delta_{i1})d_{i-1\ j}$$

$$+\left[\frac{Ebc}{2}\frac{(\pi j)^2}{L}+\frac{G_B bL}{2t}(2-\delta_{i1}-\delta_{in})\right]d_{ij}$$

$$-\left[-\frac{G_B bL}{2t}\right](1-\delta_{in})d_{i+1\ j}=0 \qquad i=1,2,\ldots,n$$

$$j=1,2,\ldots,\infty \tag{14}$$

In matrix form, equations (13) and (14) become, respectively,

$$[A_1]_j\,(a_i)_j+[A_2]_j\,(d_i)_j=(C_1)_j \qquad j=1,2,\ldots,\infty \tag{15}$$

$$[A_3]_j\,(a_i)_j+[A_4]_j\,(d_i)_j=0 \qquad j=1,2,\ldots,\infty \tag{16}$$

The matrices $[A_1]_j$, $[A_2]_j$, $[A_3]_j$, and $[A_4]_j$ are given in Appendix V.

Solution of equations (15) and (16) for (a_i) and (d_i) gives

$$(a_i)_j = [B_1]_j^{-1} (C_1)_j \qquad i = 1,2,\ldots,n \tag{17}$$

$$(d_i)_j = -[A_4]_j^{-1} \cdot [A_3]_j [B_1]_j^{-1} (C_1)_j \qquad j = 1,2,\ldots,n \tag{18}$$

where $[B_1]_j$ is given by

$$[B_1]_j = [A_1]_j - [A_2]_j \cdot [A_4]_j^{-1} \cdot [A_3]_j \qquad j = 1,2,\ldots,n \tag{19}$$

The application of equations (17), (18), and (19) to the laterally loaded beam shown in Figure 4 leads to the results plotted in Figures 5, 6, and 7.

BUCKLING UNDER AXIAL LOADING

In the beam buckling problem, the equilibrium equations (9) and the corresponding boundary condition equations (10) are all satisfied by the displacement functions

$$w_i = a_i \sin \frac{\pi x}{L} \qquad i = 1,2,\ldots,n \tag{20}$$

$$u_i = d_i \cos \frac{\pi x}{L} \qquad i = 1,2,\ldots,n \tag{21}$$

Substitution of equations (20) and (21) into the equilibrium equations (9) leads to two sets of n algebraic equations as shown in Appendix V. In matrix form these equations are

$$[A_5]\ (a_i) + [A_6]\ (d_i) = 0 \tag{22}$$

$$[A_7]\ (a_i) + [A_8]\ (d_i) = 0 \tag{23}$$

The matrices $[A_5]$, $[A_6]$, $[A_7]$, and $[A_8]$ are given in Appendix V. Solving equation (23) for (d_i) and substituting into equation (22) gives

$$[B_2]\ (a_i) = 0 \tag{24}$$

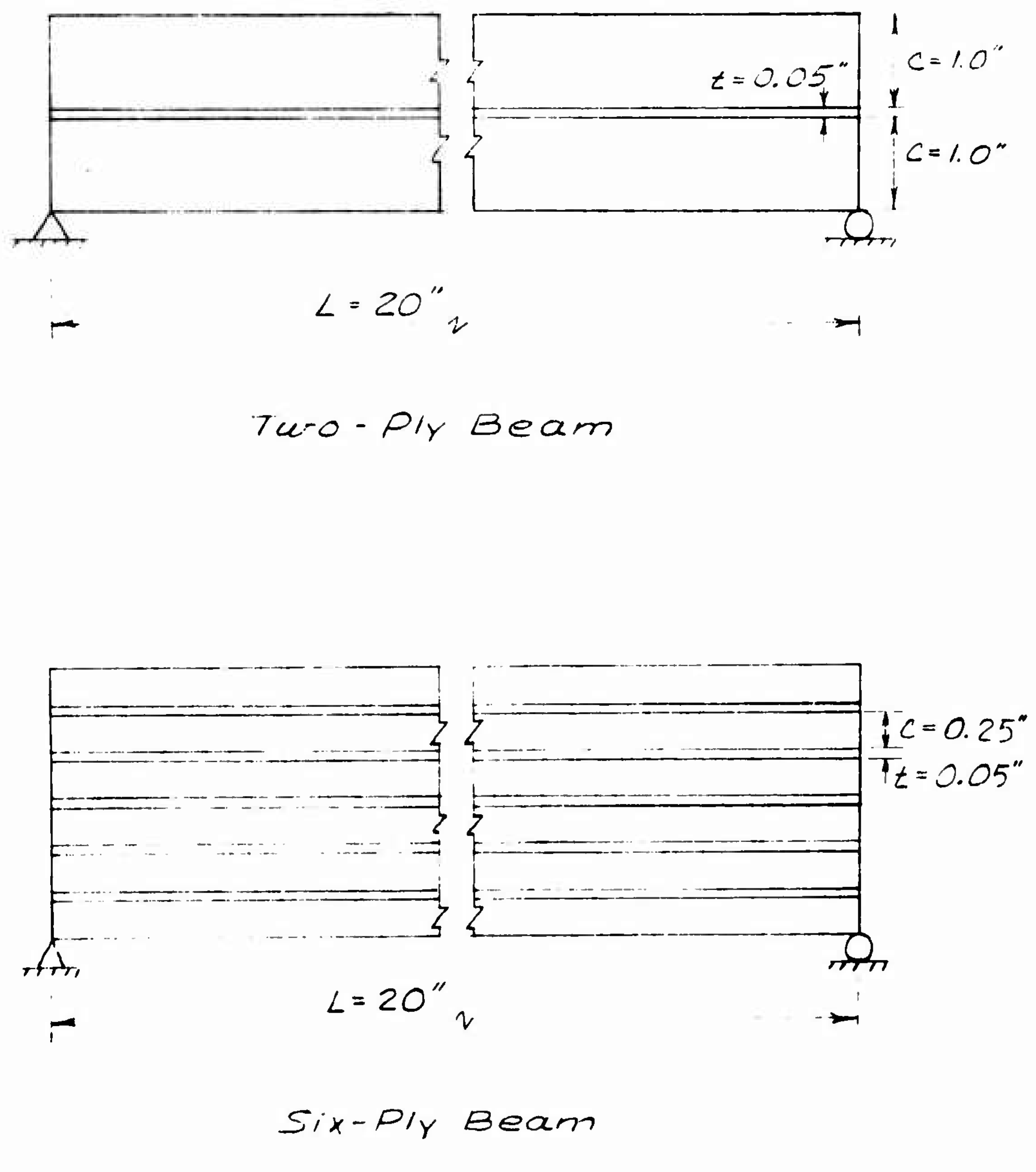

Figure 5. Laminated Beam Geometries Utilized in Bending and Buckling Solutions.

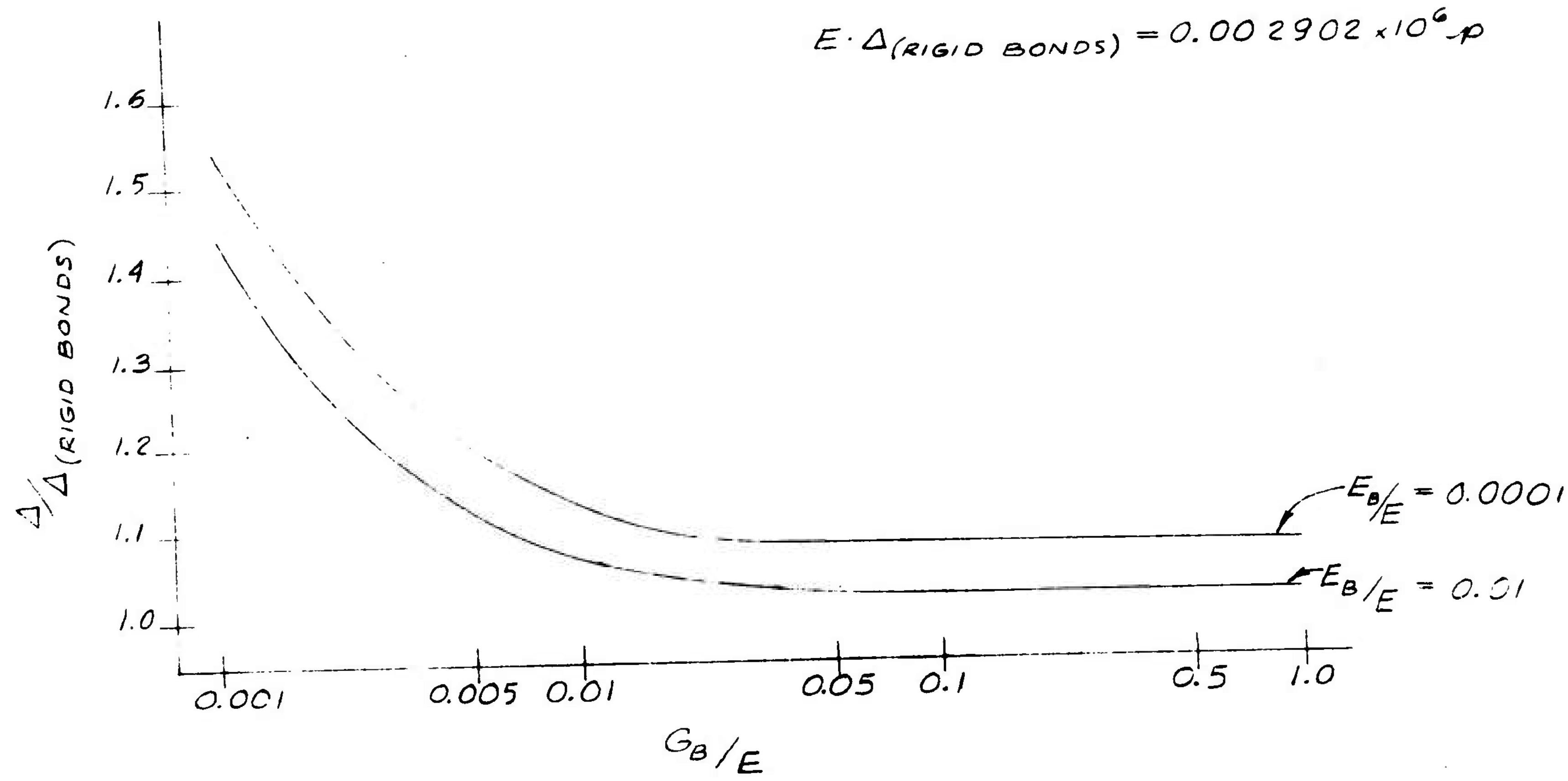

Figure 6. Maximum Deflection of Two-Ply Beam Under Uniform Loading.

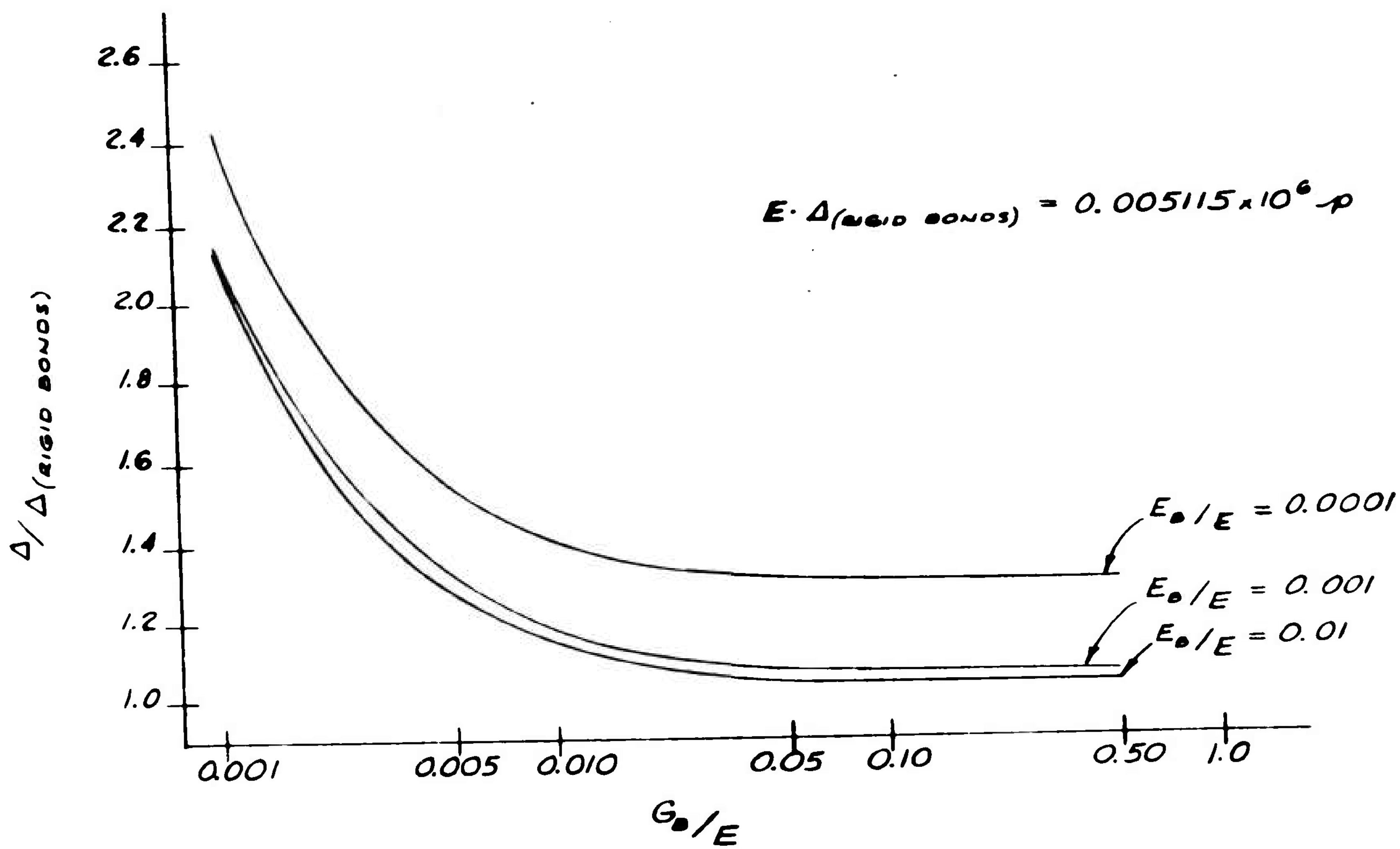

Figure 7. Maximum Deflection of Six-Ply Beam Under Uniform Loading.

where

$$[B_2] = [A_5] - [A_6][A_8]^{-1}[A_7] \tag{25}$$

For nontrivial solutions of (a_i) to exist, $[B_2]$ must vanish. Thus the eigenvalues of $[B_2]$ give the buckling values.

DISCUSSION OF RESULTS

The equations developed in this report for the bending and buckling of laminated beams have been applied to two specific constructions, one a two-ply and the other a six-ply section as shown in Figure 5. The curves plotted in Figures 6, 7, and 8 and the data in the table on page 23 have been derived from the bending solutions. The curves plotted in Figure 9 have been developed from the buckling solutions.

Figures 6 and 7 show the effects of variations of the bond elastic properties on the beam deflections for the two-ply and six-ply beams, respectively. The deflection values have been nondimensionalized with respect to the deflection obtained when the bond layers are infinitely rigid in shear and compression normal to the bending axis. As shown in Figures 6 and 7, the beam deflections are significantly affected by both the shear modulus and the extensional modulus of the bond layers. For low values of the bond shear modulus, the laminae tend to deflect as individual beams and, hence, the deflection increases. For low values of the bond extensional modulus, the bond layers tend to compress, causing additional lateral deflection of the beam. There appears to be little quantitative interaction between these two effects, however, as the curves in Figures 6 and 7 are essentially parallel.

The shear stress in the bond layers, shown in Figure 8, is found to depend essentially on the shear modulus of the bond layers and to be relatively unaffected by variations in the bond extensional modulus. The shear stresses plotted in Figure 8 have been nondimensionalized with respect to the shear stress of a beam with infinitely rigid bond layers. This is, of course, the shear stress of elementary beam theory, $\tau = VQ/Ib$. As shown in Figure 8, the actual bond shear stress does not vary significantly from the shear stress of elementary beam theory for reasonable values of the bond shear modulus. Furthermore, it is useful to note that the elementary beam theory shear stress is conservative. Hence, for design purposes, the elementary beam theory shear stress may be used.

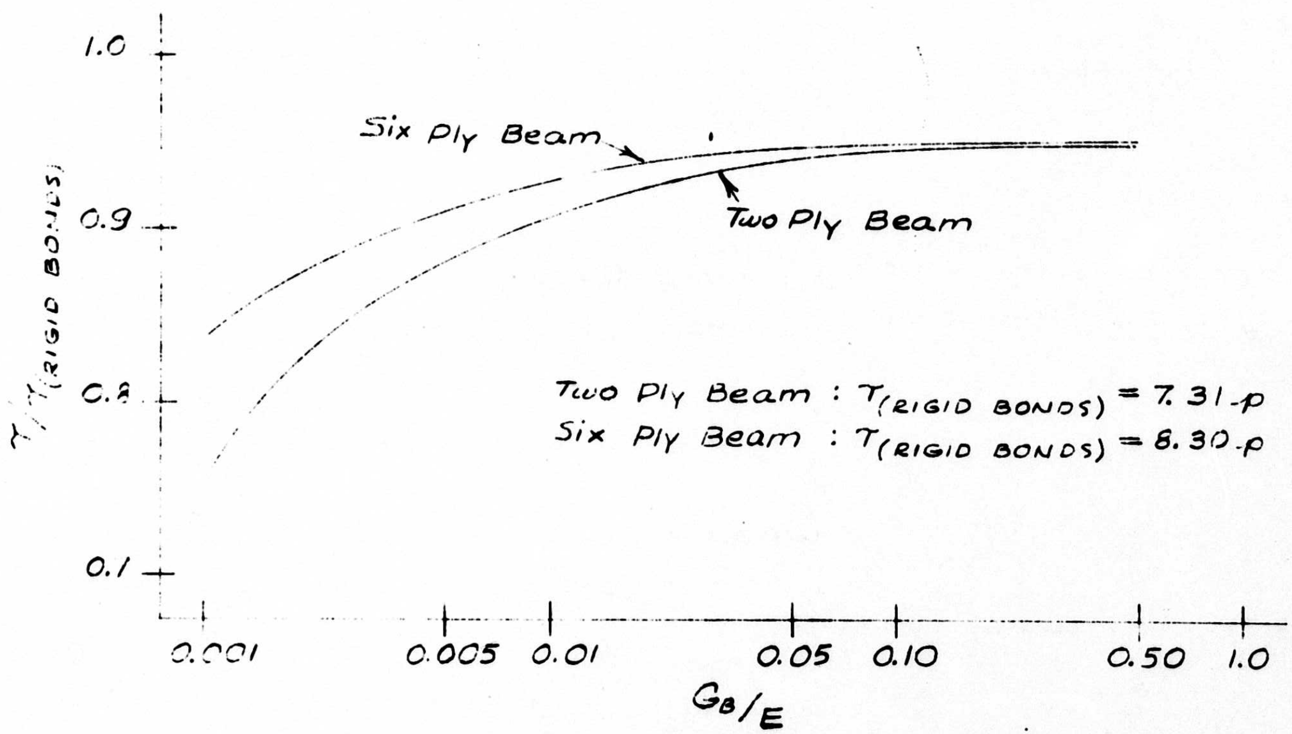

Figure 8. Maximum Shear Stress in Laminated Beams.

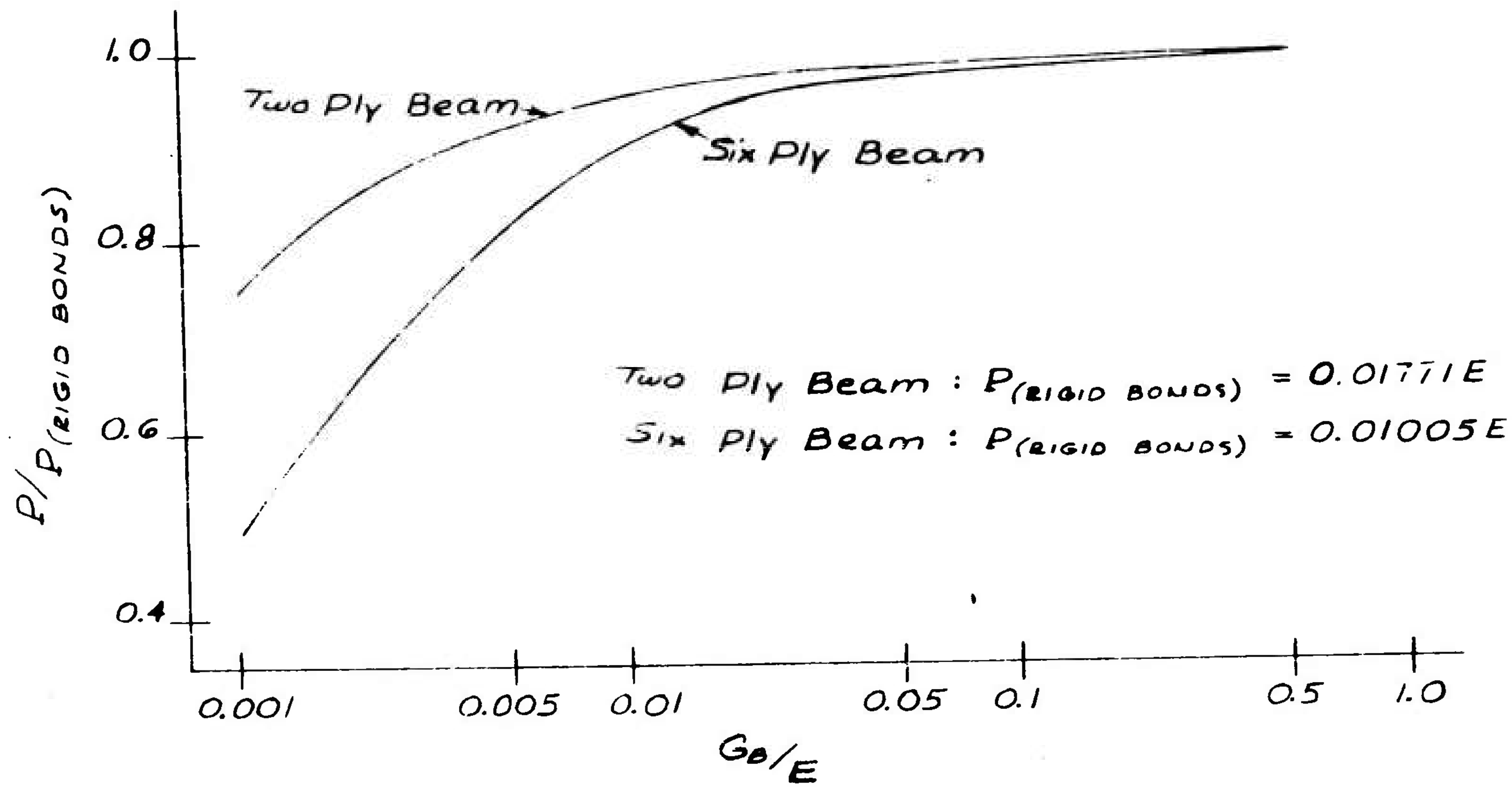

Figure 9. Buckling Loads for Laminated Beams.

The extensional stresses in the bond layers due to bending are given in the table below for the beams studied.

MAXIMUM DIRECT STRESS IN BEAMS		
E_B/E	$\sigma_{max}/\tilde{\sigma}$ *	
	Two-Ply	Six-Ply
0.001	0.00525	0.00800
0.001	0.00645	0.00884
0.1	0.00741	0.00916

* $\tilde{\sigma}$, max bending stress in beam with rigid bonds

The extensional stresses in the bond layers are found to be relatively unaffected by variations in the bond shear modulus. Also, as shown in the table, the direct stress is not greatly affected by variations in the extensional bond modulus.

The buckling loads of the laminated beams studied are found to depend essentially on the bond shear modulus, with variations in the bond extensional modulus having virtually no effect. The buckling loads presented in Figure 9 have been nondimensionalized with respect to the buckling load of a beam with rigid bond layers. As shown in Figure 9, the buckling loads are significantly affected by variations in the bond shear modulus; in fact, for beams of low bond shear modulus, the laminae tend to buckle individually. It is of interest to note that although only one deflected shape is assumed, as given by equations (20) and (21), the buckling solution yields n eigenvalues. This is attributed to the fact that within the restriction of the deflected shape implicit in (20) and (21), the beam can still develop n-mode shapes. That is, separation can occur at one interval bond layer with the result that the laminae on each side of the separation move in opposite directions. Two possible modes for a four-ply beam are shown in Figure 10. However, for reasonable values of the extensional bond modulus (high peel strength), the eigenvalues corresponding to the mode shapes shown in Figure 10 are precluded.

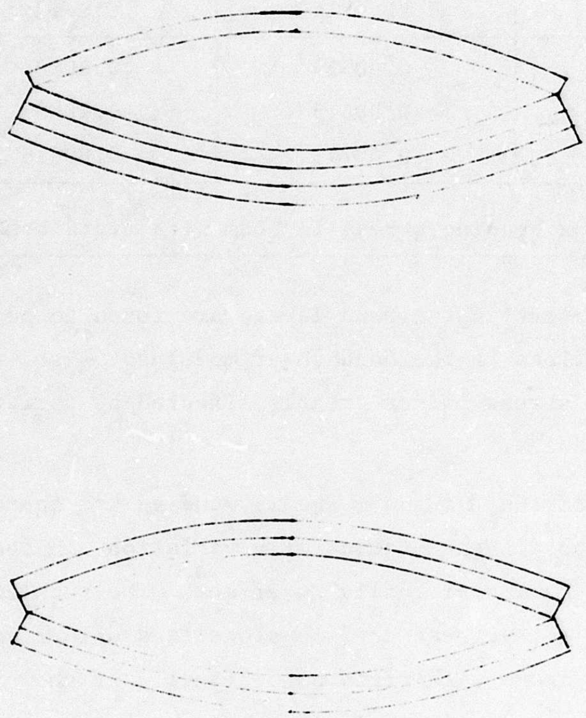

Figure 10. Two Possible Buckling Modes for a Four-Ply Laminated Beam.

CONCLUDING REMARKS

The analysis presented herein is the first step toward development of a general theory and solution procedure for laminated plates and shells. The distinguishing feature of the theory is that the transverse shear and extensional flexibilities of the bond layer between two laminae are included. The extension of the present governing equations and boundary conditions to plate and shell elements is straightforward with the incorporation of the appropriate two-dimensional strain-displacement relations in the variational principle. The quantitative results obtained for the bending and buckling of two- and six-ply beams reflect significant effects of bond flexibility; also, they indicate that redistribution and amplification of the bond stresses would occur under conditions wherein the laminae are not permitted to move relative to one another at the beam boundaries. This certainly is the case to be expected in practice. Thus, further effort should be expended on the boundary problem as well as on developing both qualitative and quantitative information for laminated plates and shells. In conclusion, it should be noted that two of the basic assumptions on which the present analysis is based are that all of the laminae and all of the bond layers possess the same elastic and geometric properties. However, the analysis presented herein is easily modified for beams with nonuniform bond layers and/or nonuniform laminae. The nonzero elements in the matrices given in Appendixes IV and V can be modified to accommodate property changes between the laminae and the bond layers.

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APPENDIX I

STRAIN-DISPLACEMENT RELATIONS FOR BOND LAYERS AND STRAIN ENERGY IN BOND LAYERS

The strain-displacement relationships for bond shear and extensional strains normal to the bending axis are developed in this appendix. The consideration of two adjacent laminae with their connecting bond layer and the distortion of a typical cross-section a-a (as shown in Figure 2) leads to the displacement of a point on the upper edge of the bond layer due to bending given by

$$\delta_1 = u_i - \frac{c}{2}\frac{dw_i}{dx} \tag{26}$$

At the lower edge of the bond, a point originally on section a-a is displaced a distance

$$\delta_2 = u_{i+1} + \frac{c}{2}\frac{dw_{i+1}}{dx} \tag{27}$$

Thus, if it is assumed that the angle of distortion of the bond layer is constant across the thickness, then this angle of distortion is

$$\alpha = \frac{\delta_1 - \delta_2}{t} = \frac{1}{t}\left[u_1 - u_{i+1} - \frac{c}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right)\right] \tag{28}$$

The angle α includes two effects. One is the shear distortion of the bond layer and the other is the slope of the bond layer due to bending. Under the assumption that bending resistance of the bond layer is negligible, this latter effect is, quantitatively,

$$\alpha_1 = \frac{1}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right) \tag{29}$$

Thus, the angle α can be written as

$$\alpha = \gamma_B + \alpha_1 = \gamma_B + \frac{1}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right) \tag{30}$$

Use of equation (28) gives

$$\gamma_B = \frac{u_i - u_{i+1}}{t} - \frac{1}{2}(1 + \frac{c}{t})\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right) \tag{31}$$

For thin bond layers the extensional strain normal to the bending axis can be assumed constant. Thus, this strain component is

$$\epsilon_z = \frac{w_{i+1} - w_i}{t} \tag{32}$$

The strain energy in the bond layers is then

$$U_B = \sum_{i=1}^{n-1} \left\{ \frac{G_B b}{2t} \int_0^L \left[u_i - u_{i+1} - \frac{c+t}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right)\right]^2 dx \right.$$

$$\left. + \frac{E_B b}{2t} \int_0^L [w_{i+1} - w_i]^2 dx \right\} \tag{33}$$

APPENDIX II

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS FOR BENDING UNDER DISTRIBUTED LOADING

For the bending problem shown in Figure 3, the potential of the applied loads is

$$V_p = -\int_0^L pbw_1 \, dx \tag{34}$$

The strain energy associated with bending and stretching of the laminae is

$$U_3 = \sum_{i=1}^{n} \left\{ \frac{Ebc}{2} \int_0^L \left(\frac{du_i}{dx}\right)^2 dx + \frac{EI_o}{2} \int_0^L \left(\frac{d^2w_i}{dx^2}\right)^2 dx \right\} \tag{35}$$

From equations (33), (34), and (35), the total potential functional is

$$U + V_p = \sum^{n} \left\{ \frac{Ebc}{2} \int_0^L \left(\frac{du_i}{dx}\right)^2 dx + \frac{EI_o}{2} \int_0^L \left(\frac{d^2w_i}{dx^2}\right)^2 dx \right\}$$

$$+ \sum_{i=1}^{n-1} \left\{ \frac{G_Bb}{2t} \int_0^L \left[u_i - u_{i+1} - \frac{c+t}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right)\right]^2 dx \right.$$

$$\left. + \frac{E_Bb}{2t} \int_0^L [w_{i+1} - w_i]^2 dx \right\} - \int_0^L p \cdot b \cdot w_1 \cdot dx \tag{36}$$

The variation with respect to w_i gives

$$\delta_{w_i}(U+V_p) = EI_o \int_0^L \frac{d^2w_i}{dx^2} \cdot \frac{d^2\delta w_i}{dx^2} dx$$

$$+(1-\delta_{i1}-\delta_{in}) \frac{G_Bb}{2} \cdot \frac{c+t}{t} \int_0^L \left[u_{i+1} - u_{i-1} + \frac{c+t}{2}\left(\frac{dw_{i-1}}{dx} + \frac{dw_{i+1}}{dx}\right)\right]$$

(continued)

$$\cdot \frac{d}{dx}\,\delta w_i\, dx + (1-\delta_{i1}-\delta_{in})\,\frac{E_B b}{t}\int_0^L (-w_{i-1} + 2w_i - w_{i+1})\,\delta w_i\, dx$$

$$+(\delta_{i1})\,\frac{G_B b}{2}\cdot\frac{c+t}{2t}\int_0^L \left[u_2 - u_1 + \frac{c+t}{2}\left(\frac{dw_1}{dx} + \frac{dw_2}{dx}\right)\right]\frac{d\delta w_i}{dx}\, dx$$

$$+(\delta_{i1})\,\frac{E_B b}{t}\int_0^L [w_1 - w_2]\,\delta w_i\, dx$$

$$+(\delta_{in})\,\frac{G_B b}{2}\cdot\frac{c+t}{2t}\int_0^L \left[u_n - u_{n-1} + \frac{c+t}{2}\left(\frac{dw_{n-1}}{dx} + \frac{dw_n}{dx}\right)\right]\frac{d\delta w_n}{dx}\, dx$$

$$+(\delta_{in})\,\frac{E_B b}{t}\int_0^L [w_n - w_{n-1}]\,\delta w_n\, dx - (\delta_{i1})\int_0^L bp\,\delta w_1\, dx = 0$$

$$i = 1,2,\ldots,n \qquad (37)$$

Integration by parts leads to the set of n coupled, differential equations and $2n$ associated boundary conditions giver. by

$$EI_o\,\frac{d^4 w_i}{dx^4} - (1-\delta_{i1}-\delta_{in})\,\frac{E_B b}{t}\,(-w_{i-1} + 2w_i - w_{i+1}) + (\delta_{i1})\,\frac{E_B b}{t}(w_1 - w_2)$$

$$+(\delta_{in})\,\frac{E_B b}{t}(w_n - w_{n-1}) - (\delta_{in})\,\frac{G_B b(c+t)}{2t}\left[\frac{du_n}{dx} - \frac{du_{n-1}}{dx}\right.$$

$$\left. + \frac{c+t}{2}\left(\frac{d^2 w_n}{dx^2} + \frac{d^2 w_{n-1}}{dx^2}\right)\right] - (\delta_{i1})\,\frac{G_B b(c+t)}{2t}\left[\frac{du_2}{dx} - \frac{du_1}{dx}\right.$$

$$\left. + \frac{c+t}{2}\left(\frac{d^2 w_2}{dx^2} + \frac{d^2 w_1}{dx^2}\right)\right] - (1-\delta_{i1}-\delta_{in})\,\frac{G_B b(c+t)}{2t}\left[\frac{du_{i+1}}{dx} - \frac{du_{i-1}}{dx}\right.$$

$$\left. + \frac{c+t}{2}\left(\frac{d^2 w_{i-1}}{dx^2} + 2\,\frac{d^2 w_i}{dx^2} + \frac{d^2 w_{i+1}}{dx^2}\right)\right] - (\delta_{i1})p = 0 \quad i = 1,2,\ldots,n \qquad (38)$$

$$\left\{-EI_o \frac{d^3 w_i}{dx^3} + (\delta_{i1}) \frac{G_B b(c+t)}{2t}\left[u_2 - u_1 + \frac{c+t}{2}\left(\frac{dw_1}{dx} + \frac{dw_2}{dx}\right)\right]\right.$$

$$+(1-\delta_{i1}-\delta_{in}) \frac{G_B b(c+t)}{2t}\left[u_{i+1} - u_{i-1} + \frac{c+t}{2}\left(\frac{dw_{i+1}}{dx} + 2\frac{dw_i}{dx} + \frac{dw_{i-1}}{dx}\right)\right]$$

$$\left. + \delta_{in} \frac{G_B b(c+t)}{2t}\left[u_n - u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right]\right\}_0^L = 0$$

or

$$\delta w_i \Big|_0^L = 0 \qquad i = 1,2,\ldots,n \tag{39}$$

$$EI_o \frac{d^2 w_i}{dx^2}\Big|_0^L = 0 \qquad \text{or} \qquad \frac{d}{dx}\delta w_i \Big|_0^L = 0 \qquad i = 1,2,\ldots,n \tag{40}$$

The variation with respect to u_i gives

$$\delta_{u_i}(U+V_p) = Ebc \int_0^b \frac{du_i}{dx} \cdot \frac{d\delta u_i}{dx} \cdot dx$$

$$+(1-\delta_{i1}-\delta_{in}) \frac{G_B b}{t} \int_0^L \left[-u_{i-1} + 2u_i - u_{i+1} + \frac{c+t}{2}\left[\left(\frac{dw_{i-1}}{dx} + \frac{dw_{i+1}}{dx}\right)\right]\right] \delta u_i dx$$

$$+(\delta_{i1}) \frac{G_B b}{t} \int_0^L \left[u_1 - u_2 - \frac{c+t}{2}\left(\frac{dw_1}{dx} + \frac{dw_2}{dx}\right)\right] \delta u_1 dx$$

$$+(\delta_{in}) \frac{G_B b}{t} \int_0^L \left[u_n - u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right] \delta u_n dx = 0$$

$$i = 1,2,\ldots,n \tag{41}$$

Integration by parts leads to the set of n coupled, differential equations and the associated set of n boundary conditions given by

$$
\begin{aligned}
&-Ebc\,\frac{d^2u_i}{dx^2} - (\delta_{i1})\,\frac{G_Bb}{t}\left[u_2-u_1 + \frac{c+t}{2}\left(\frac{du_2}{dx} + \frac{dw_1}{dx}\right)\right] \\
&\quad +(1-\delta_{i1}-\delta_{in})\,\frac{G_Bb}{t}\left[-u_{i-1} + 2u_i-u_{i+1} + \frac{c+t}{2}\left(\frac{dw_{i-1}}{dx} - \frac{dw_{i+1}}{dx}\right)\right] \\
&\quad +(\delta_{in})\,\frac{G_Bb}{t}\left[u_n-u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right] = 0 \\
&\qquad\qquad i = 1,2,\ldots,n \qquad (42)
\end{aligned}
$$

$$
Ebc\,\frac{du_i}{dx}\bigg|_0^L = 0 \qquad \text{or} \qquad \delta u_i\bigg|_0^L = 0 \qquad i = 1,2,\ldots,n \qquad (43)
$$

APPENDIX III

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS FOR BUCKLING UNDER AXIAL LOADING

For the buckling problem shown in Figure 4, the potential of the applied loads is

$$V_P = -\sum_{i=1}^{n} \frac{P}{n} \int_0^L \frac{1}{2}\left(\frac{dw_i}{dx}\right)^2 dx \tag{44}$$

With the use of equations (33) in Appendix I and equation (35) in Appendix II, the total potential functional for buckling of laminated beams under axial loading is then

$$U + V_P = \sum_{i=1}^{n} \left\{ \frac{Ebc}{2t} \int_0^L \left(\frac{du_i}{dx}\right)^2 dx + \frac{EI_o}{2} \int_0^L \left(\frac{d^2 w_i}{dx^2}\right)^2 dx \right\}$$

$$+ \sum_{i=1}^{n-1} \left\{ \frac{G_B b}{2t} \int_0^L \left[u_i - u_{i-1} - \frac{c+t}{2}\left(\frac{dw_i}{dx} + \frac{dw_{i+1}}{dx}\right)\right]^2 dx \right.$$

$$\left. + \frac{E_B b}{2t} \int_0^L [w_{i+1} - w_i]^2 dx \right\} - \sum_{i=1}^{n} \left\{ \frac{P}{n} \int_0^L \frac{1}{2}\left(\frac{dw_i}{dx}\right)^2 dx \right\} \tag{45}$$

The variation with respect to w_i gives

$$\delta_{w_i}(U+V_P) = EI_o \int_0^L \frac{d^2 w_i}{dx^2} \cdot \frac{d^2 \delta w_i}{dx^2} dx$$

$$+(1-\delta_{i1}-\delta_{in}) \frac{G_B b}{2} \cdot \frac{c+t}{t} \int_0^L \left[u_{i+1} - u_{i-1} + \frac{c+t}{2}\left(\frac{dw_{i-1}}{dx} + \frac{dw_{i+1}}{dx}\right)\right]$$

$$\cdot \frac{d}{dx} \delta w_i dx + (1-\delta_{i1}-\delta_{in}) \frac{E_B b}{t} \int_0^L (-w_{i-1} + 2w_i - w_{i+1}) \delta w_i dx$$

(continued)

$$+(\delta_{i1})\,\frac{G_B b}{2}\cdot\frac{c+t}{2t}\int_0^L\left[u_2-u_1+\frac{c+t}{2}\left(\frac{dw_1}{dx}+\frac{dw_2}{dx}\right)\right]\frac{d\delta w_1}{dx}\,dx$$

$$+(\delta_{i1})\,\frac{E_B b}{t}\int_0^L[w_1-w_2]\delta w_1 dx$$

$$+(\delta_{in})\,\frac{G_B b}{2}\cdot\frac{c+t}{2t}\int_0^L\left[u_n-u_{n-1}+\frac{c+t}{2}\left(\frac{dw_{n-1}}{dx}+\frac{dw_n}{dx}\right)\right]\frac{d\delta w_n}{dx}\,dx$$

$$+(\delta_{in})\,\frac{E_B b}{t}\int_0^L[w_n-w_{n-1}]\delta w_n\,dx$$

$$-\frac{P}{n}\int_0^L\frac{dw_i}{dx}\cdot\frac{d\delta w_i}{dx}\,dx=0 \qquad i=1,2,\ldots,n \qquad (46)$$

Integration of equation (46) by parts leads to the set of n coupled, differential equations and the $2n$ associated boundary conditions given by

$$EI_o\frac{d^4w_i}{dx^4}-(1-\delta_{i1}-\delta_{in})\,\frac{E_B b}{t}(-w_{i-1}+2w_i-w_{i+1})+(\delta_{i1})\,\frac{E_B b}{t}(w_1-w_2)$$

$$+(\delta_{in})\,\frac{E_B b}{t}(w_n-w_{n-1})-(\delta_{in})\,\frac{G_B b(c+t)}{2t}\left[\frac{du_n}{dx}-\frac{du_{n-1}}{dx}\right.$$

$$\left.+\frac{c+t}{2}\left(\frac{d^2w_n}{dx^2}+\frac{d^2w_{n-1}}{dx^2}\right)\right]-(\delta_{i1})\,\frac{G_B b(c+t)}{2t}\left[\frac{du_2}{dx}-\frac{du_1}{dx}\right.$$

$$\left.+\frac{c+t}{2}\left(\frac{d^2w_2}{dx^2}+\frac{d^2w_1}{dx^2}\right)\right]-(1-\delta_{i1}-\delta_{in})\,\frac{G_B b(c+t)}{2t}\left[\frac{du_{i+1}}{dx}-\frac{du_{i-1}}{dx}\right.$$

$$\left.+\frac{c+t}{2}\left(\frac{d^2w_{i-1}}{dx^2}+2\,\frac{d^2w_i}{dx^2}+\frac{d^2w_{i+1}}{dx}\right)\right]+\frac{P}{n}\,\frac{d^2w_i}{dx^2}=0 \qquad i=1,2,\ldots,n \qquad (47)$$

$$EI_o \frac{d^2 w_i}{dx^2}\Bigg|_0^L = 0 \qquad \text{or,} \qquad \frac{d}{dx}\,\delta w_i\Bigg|_0^L \qquad\qquad i = 1,2,\ldots,n \qquad (48)$$

$$\Bigg\{-EI_o \frac{d^3 w_i}{dx^3} + (\delta_{i1})\,\frac{G_B b(c+t)}{2t}\left[u_2 - u_1 + \frac{c+t}{2}\left(\frac{dw_1}{dx} + \frac{dw_2}{dx}\right)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\,\frac{G_B b(c+t)}{2t}\left[u_{i+1} - u_{i-1} + \frac{c+t}{2}\left(\frac{dw_{i+1}}{dx} + 2\,\frac{dw_i}{dx} + \frac{dw_{i-1}}{dx}\right)\right]$$

$$+(\delta_{in})\,\frac{G_B b(c+t)}{2t}\left[u_n - u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right] - \frac{P}{n}\,\frac{dw_i}{dx}\Bigg\}_0^L = 0$$

or

$$\delta w_i\Bigg|_0^L = 0 \qquad\qquad i = 1,2,\ldots,n \qquad (49)$$

The variation of equation (45) with respect to u_i gives

$$\delta_{u_i}(U+V_P) = Ebc\int_0^b \frac{du_i}{dx}\cdot\frac{d\delta u_i}{dx}\cdot dx$$

$$+(1-\delta_{i1}-\delta_{in})\cdot\frac{G_B b}{t}\int_0^L\left[-u_{i-1}+2u_i-u_{i+1} + \frac{c+t}{2}\left(\frac{dw_{i-1}}{dx} + \frac{dw_{i+1}}{dx}\right)\right]\delta u_i\,dx$$

$$+(\delta_{i1})\cdot\frac{G_B b}{t}\int_0^L\left[u_1-u_2 - \frac{c+t}{2}\left(\frac{dw_1}{dx} + \frac{dw_2}{dx}\right)\right]\delta u_1\,dx$$

$$+(\delta_{in})\cdot\frac{G_B b}{t}\int_0^L\left[u_n-u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right]\delta u_n\,dx$$

$$i = 1,2,\ldots,n \qquad (50)$$

Integration of equation (50) by parts gives the set of n coupled, differential equations and n associated boundary conditions given by

$$-Ebc\,\frac{d^2u_i}{dx^2} - (\delta_{i1})\,\frac{G_B b}{t}\left[u_2 - u_1 + \frac{c+t}{2}\left(\frac{dw_2}{dx} + \frac{dw_1}{dx}\right)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\,\frac{G_B b}{t}\left[-u_{i-1} + 2u_i - u_{i+1} + \frac{c+t}{2}\left(-\frac{dw_{i-1}}{dx} - \frac{dw_{i+1}}{dx}\right)\right]$$

$$+(\delta_{in})\,\frac{G_B b}{t}\left[u_n - u_{n-1} + \frac{c+t}{2}\left(\frac{dw_n}{dx} + \frac{dw_{n-1}}{dx}\right)\right] = 0$$

$$i = 1,2,\ldots,n \qquad (51)$$

$$Ebc\,\frac{du_i}{dx}\Bigg|_0^L = 0 \qquad \text{or} \qquad \delta u_i\Bigg|_0^L = 0 \qquad i = 1,2,\ldots,n \qquad (52)$$

APPENDIX IV

SOLUTION FOR BENDING OF A LAMINATED BEAM BY A UNIFORMLY DISTRIBUTED LOAD

The boundary conditions for this problem, as given by equations (39), (40), and (43) in Appendix II, all are satisfied by the complete displacement functions

$$w_i = \sum_{i=1}^{n} \sum_{j=1,3}^{\infty} a_{ij} \sin \frac{j\pi x}{L} \tag{53}$$

$$u_i = \sum_{i=1}^{n} \sum_{j=1,3}^{\infty} d_{ij} \cos \frac{j\pi x}{L} \tag{54}$$

Substitution of expressions (53) and (54) into the total potential functional, equation (36) in Appendix II, gives

$$\begin{aligned}
U+V_p = \sum_{i=1}^{n} \Bigg\{ & \frac{Ebc}{2t} \int_0^L \left[\sum_{j=1,3}^{\infty} \left(\frac{j\pi}{L} d_{ij} \sin \frac{j\pi x}{L} \right) \right]^2 dx \\
& + \frac{EI_o}{2} \int_0^L \left[\sum_{j=1,3}^{\infty} -\left(\frac{j\pi}{L}\right)^2 a_{ij} \sin \frac{j\pi x}{L} \right]^2 dx \Bigg\} \\
& + \sum_{i=1}^{n-1} \Bigg\{ \frac{E_B b}{2t} \int_0^L \left[\sum_{j=1,3}^{\infty} a_{i+1\ j} \sin \frac{j\pi x}{L} - \sum_{j=1,3}^{\infty} a_{ij} \sin \frac{j\pi x}{L} \right]^2 dx \Bigg\} \\
& + \sum_{i=1}^{n-1} \Bigg\{ \frac{G_B b}{2t} \int_0^L \Bigg[\sum_{j=1,3}^{\infty} d_{i+1\ j} \cos \frac{j\pi x}{L} - \sum_{j=1,3}^{\infty} d_{ij} \cos \frac{j\pi x}{L} \\
& + \frac{c+t}{2} \sum_{j=1,3}^{\infty} \frac{j\pi}{L} a_{i+1\ j} \cos \frac{j\pi x}{L} + \frac{c+t}{2} \sum_{j=1,3}^{\infty} \frac{j\pi}{L} a_{ij} \cos \frac{j\pi x}{L} \Bigg]^2 \Bigg\} dx
\end{aligned}$$

(continued)

$$- pb\int_0^L \left[\sum_{j=1,3}^{\infty} a_{ij} \sin\frac{j\pi x}{L}\right] dx \tag{55}$$

Evaluation of the integrals in equation (55) gives

$$U+V_p = \sum_{i=1}^{n}\left\{\frac{Ebc}{2}\sum_{j=1,3}^{\infty} \left(\frac{j\pi}{L}\right)^4 \frac{L}{2} d_{ij}^2 + \frac{EI_o}{2}\sum_{j=1,3}^{\infty} \left(\frac{j\pi}{L}\right)^4 \frac{L}{2} a_{ij}^2\right\}$$

$$+\sum_{i=1}^{n-1}\left\{\frac{E_B b}{2t}\sum_{j=1,3}^{\infty} \frac{L}{2} a_{i+1}^2 - \frac{E_B b}{t}\sum_{j=1,3}^{\infty} \frac{L}{2} a_{ij}a_{i+1j} + \frac{E_B b}{2t}\sum_{j=1,3}^{\infty} \frac{L}{2} a_{ij}^2\right\}$$

$$+\sum_{i=1}^{n-1}\left\{\frac{G_B b}{2t}\sum_{j=1,3}^{\infty} \frac{L}{2} d_{i+1j}^2 + \frac{G_B b}{2t}\sum_{j=1,3}^{\infty} \frac{L}{2} d_{ij}^2\right\}$$

$$+\sum_{i=1}^{n-1}\left\{\frac{G_B b}{2t}\sum_{j=1,3}^{\infty} \frac{[j\pi(c+t)]^2}{8L} a_{i+1}^2 + \frac{G_B b}{2t}\sum_{j=1,3}^{\infty} \frac{[j\pi(c+t)]^2}{8L} a_{ij}^2\right\}$$

$$+\sum_{i=1}^{n-1}\left\{-\frac{G_B b}{t}\sum_{j=1,3}^{\infty} \frac{L}{2} d_{ij}d_{i+1j} + \frac{G_B b}{t}\sum_{j=1,3}^{\infty} \frac{j\pi(c+t)}{4} a_{i+1j} d_{ij}\right\}$$

$$+\sum_{i=1}^{n-1}\left\{\frac{G_B b}{t}\sum_{j=1,3}^{\infty} \frac{j\pi(c+t)}{4} a_{ij}d_{i+1j} - \frac{G_B b}{t}\sum_{j=1,3}^{\infty} \frac{j\pi(c+t)}{4} a_{i+1j}d_{ij}\right\}$$

$$+\sum_{i=1}^{n-1}\left\{\frac{G_B b}{t}\sum_{j=1,3}^{\infty} \frac{j\pi(c+t)}{t} a_{ij}d_{ij} + \frac{G_B b}{t}\sum_{j=1,3}^{\infty} \frac{[j\pi(c+t)]^2}{8L} a_{i+1j}a_{ij}\right\}$$

$$-pb\sum_{j=1,3}^{\infty} \frac{2L}{j\pi} a_{ij} \tag{56}$$

Variation of equation (56) with respect to a_{ij} leads to

$$\delta_{a_{ij}}(U+V_p) = \left\{\frac{EI_o}{2}\frac{(j\pi)^4}{L^3} + \left[\frac{E_B bL}{2t} + \frac{G_B b}{t}\frac{[j\pi(c+t)]^2}{8L}\right](2-\delta_{i1}-\delta_{in})\right\} a_{ij}$$

(continued)

$$+\left\{-\frac{E_Bb}{2t}L+\frac{G_Bb}{t}\frac{[j\pi(c+t)]^2}{8L}\right\}(1-\delta_{i1})a_{i-1j}$$

$$+\left\{-\frac{E_BbL}{2t}+\frac{G_Bb}{t}\frac{[j\pi(c+t)]^2}{8L}\right\}(1-\delta_{in})a_{i+1j}$$

$$+\left\{-\frac{G_Bb}{t}\frac{j\pi(c+t)}{4}\right\}(1-\delta_{i1})d_{i-1j}$$

$$+\left\{\frac{G_Bb}{t}\frac{j\pi(c+t)}{4}\right\}(1-\delta_{in})d_{i+1j}+\left\{\frac{G_Bb}{t}\frac{j\pi(c+t)}{4}\right\}(\delta_{in}-\delta_{i1})d_{ij}$$

$$-\frac{2Lpb}{j\pi}(\delta_{i1})=0 \qquad i=1,2,\ldots,\infty \qquad (57)$$

Equation (57) can be written as

$$[A_5]_j\,(a_i)_j+[A_6]_j\,(d_i)_i=(C_1)_j$$

where

$$[A_5]_j=\begin{vmatrix} k_9 & k_{10} & 0 & 0 & \cdots & 0 & 0\\ k_{10} & k_{11} & k_{10} & 0 & \cdots & 0 & 0\\ 0 & k_{10} & k_{11} & k_{10} & 0 & \cdots & 0\\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & k_{10} & k_{11} & k_{10}\\ 0 & 0 & \cdots & 0 & 0 & k_{10} & k_9 \end{vmatrix} \quad (58) \qquad ,\ (C_1)_j=\begin{bmatrix}\frac{2Lpb}{j\pi}\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix} \quad (59)$$

$$[A_6]_j=\begin{vmatrix} -k_{12} & k_{12} & 0 & 0 & 0 & \cdots & 0\\ -k_{12} & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & -k_{12} & 0 & k_{12} & 0 & \cdots & 0\\ \vdots & & & & & & \\ 0 & \cdots & 0 & 0 & -k_{12} & 0 & k_{12}\\ 0 & \cdots & 0 & 0 & 0 & -k_{12} & k_{12} \end{vmatrix} \quad (60)$$

with

$$\left.\begin{aligned}
k_9 &= \frac{EI_o}{2}\frac{(j\pi)^4}{L^3} + \frac{E_BbL}{2t} + \frac{G_Bb}{t}\frac{[j\pi(c+t)]^2}{8L} \\
k_{10} &= -\frac{E_BbL}{2t} + \frac{G_Bb}{t}\frac{[j\pi(c+t)]^2}{8L} \\
k_{11} &= \frac{EI_o}{2}\frac{(j\pi)^4}{L^3} + \frac{E_BbL}{t} + \frac{G_Bb}{t}\frac{[j\pi(c+t)]^2}{4L} \\
k_{12} &= \frac{G_Bb}{t}\frac{j\pi(c+t)}{4}
\end{aligned}\right\} \qquad (61)$$

Variation of equation (56) with respect to d_{ij} yields

$$\delta_{d_{ij}}(U+V_p)\left(\frac{j\pi G_Bb(c+t)}{4t}\right)(1-\delta_{i1})\, a_{i-1j} + \left(\frac{j\pi G_Bb(c+t)}{4t}\right)(\delta_{in}-\delta_{i1})\, a_{ij}$$

$$+ \left(-\frac{j\pi G_Bb(c+t)}{4t}\right)(1-\delta_{in})a_{i+1j} + \left(-\frac{G_BbL}{2t}\right)(1-\delta_{i1})d_{i-1j}$$

$$+ \left[\frac{Ebc}{2}\frac{(\pi j)^2}{L} + \frac{G_BbL}{2t}(2-\delta_{i1}-\delta_{in})\right]d_{ij}$$

$$+ \left(-\frac{G_BbL}{2t}\right)(1-\delta_{in})d_{i+1j} = 0 \qquad i = 1,2,\ldots,n \qquad (62)$$

Equation (62) can be written as

$$[A_7]_j\,(a_i)_j + [A_8]_j\,(d_i)_j = 0 \qquad (63)$$

where

$$[A_7]_j = \begin{vmatrix}
-k_{12} & -k_{12} & 0 & 0 & 0 & \cdots & 0 \\
k_{12} & 0 & -k_{12} & 0 & 0 & \cdots & 0 \\
0 & k_{12} & 0 & -k_{12} & 0 & \cdots & 0 \\
\vdots & & & & & & \\
0 & \cdots & 0 & 0 & k_{12} & 0 & -k_{12} \\
0 & \cdots & 0 & 0 & 0 & k_{12} & k_{12}
\end{vmatrix} \qquad (64)$$

$$[A_8]_j = \begin{vmatrix} k_{13} & -k_{14} & 0 & 0 & \cdots & 0 & 0 \\ -k_{14} & k_{15} & -k_{14} & 0 & \cdots & 0 & 0 \\ 0 & -k_{14} & k_{15} & -k_{14} & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & 0 & -k_{14} & k_{15} & -k_{14} \\ C & \cdots & 0 & 0 & 0 & -k_{14} & k_{13} \end{vmatrix} \tag{65}$$

and

$$\left.\begin{aligned} k_{13} &= \frac{Ebc}{2L}(\pi j)^2 + \frac{G_B bL}{2t} \\ k_{14} &= \frac{G_B bL}{2t} \\ k_{15} &= \frac{Ebc(j\pi)^2}{2L} + \frac{G_B bL}{2t} \end{aligned}\right\} \tag{66}$$

APPENDIX V

SOLUTION FOR BUCKLING OF A LAMINATED BEAM UNDER AXIAL LOADING

All of the governing equations and boundary conditions developed in Appendix III (equations (47), (48), (49), (51) and (52)) are satisfied by the displacement functions

$$w_i = a_i \sin \frac{\pi x}{L} \qquad i = 1,2,\ldots,n \tag{67}$$

$$u_i = d_i \cos \frac{\pi x}{L} \qquad i = 1,2,\ldots,n \tag{68}$$

Substitution of the functions (67) and (68) into the set of differential equations (47) as given in Appendix III leads to

$$\frac{\pi^4}{L^4} EI_o\, a_i - (1-\delta_{in}) \frac{E_B b}{t}(a_{i+1} - a_i)$$

$$- (\delta_{in}) \frac{G_B b(c+t)}{2t}\left[-\frac{\pi}{L} d_n + \frac{\pi}{L} d_{n-1} - \frac{\pi^2}{L^2} \frac{c+t}{2} (a_n + a_{n-1})\right]$$

$$- (1-\delta_{i1}-\delta_{in}) \frac{G_B b(c+t)}{2t}\left[- \frac{\pi}{L}(d_{i+1}-d_{i-1})- \frac{\pi^2}{L^2} \frac{c+t}{2}(a_{i-1}+2a_i+a_{i+1})\right]$$

$$- (\delta_{i1}) \frac{G_B b(c+t)}{2t} \left[- \frac{\pi}{L}(d_2-d_1)- \frac{\pi^2}{L^2} \frac{c+t}{2}(a_2+a_1)\right] - \frac{\pi^2}{L^2} \frac{P}{n} w_i = 0$$

$$i = 1,2,\ldots,n \tag{69}$$

The set of equations (69) can be written in the matrix form

$$[A_1]\,(a_i) + [A_2]\,(d_i) = 0 \tag{70}$$

where

$$[A_1] = \begin{vmatrix} k_1 & k_2 & 0 & 0 & \cdots & 0 & 0 \\ k_2 & k_3 & k_2 & 0 & \cdots & 0 & 0 \\ 0 & k_2 & k_3 & k_2 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & 0 & k_2 & k_3 & k_1 \\ 0 & \cdots & 0 & 0 & 0 & k_2 & k_1 \end{vmatrix} \tag{71}$$

$$[A_2] = \begin{vmatrix} -k_4 & k_4 & 0 & 0 & \cdots & 0 & 0 \\ -k_4 & 0 & k_4 & 0 & \cdots & 0 & 0 \\ 0 & -k_4 & 0 & k_4 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & 0 & -k_4 & 0 & k_4 \\ 0 & 0 & 0 & 0 & 0 & k_4 & -k_4 \end{vmatrix} \tag{72}$$

and

$$\left.\begin{aligned} k_1 &= \frac{\pi^4}{L^4} EI_o + \frac{E_B b}{2t} + \frac{\pi^2}{L^2} \frac{G_B b(c+t)^2}{4t} - \frac{\pi^2}{L^2} \frac{P}{n} \\ k_2 &= \frac{\pi^2}{L^2} \frac{G_B b(c+t)^2}{4t} - \frac{E_B b}{2t} \\ k_3 &= \frac{\pi^4}{L^4} EI_o + \frac{E_B b}{t} + \frac{\pi^2}{L^2} \frac{G_B b(c+t)}{2t} - \frac{\pi^2}{L^2} \frac{P}{n} \\ k_4 &= \frac{\pi}{L} \frac{G_B b(c+t)}{2t} \end{aligned}\right\} \tag{73}$$

Substitution of the displacement functions (67) and (68) into the set of equations (51) in Appendix III yields

$$\frac{\pi^2}{L^2}\,\mathrm{Ebc}\; b_i \; -(\delta_{i1})\,\frac{G_B b}{t}\left[d_2 - d_1 + \frac{\pi}{L}\,\frac{c+t}{2}(a_2 + a_1)\right]$$

$$+(1-\delta_{i1}-\delta_{in})\,\frac{G_B b}{t}\left[-d_{i-1} + 2d_i - d_{i+1} + \frac{\pi}{L}\,\frac{c+t}{2}(a_{i-1} - a_{i+1})\right]$$

$$+\,\delta_{in}\,\frac{G_B b}{t}\left[d_n - d_{n-1} + \frac{\pi}{L}\,\frac{c+t}{2}(a_n - a_{n-1})\right] = 0$$

$$i = 1,2,\ldots,n \tag{74}$$

The set of equations (74) can be written in matrix form as

$$[A_3]\,(a_i) + [A_4]\,(d_i) = 0 \tag{75}$$

where

$$[A_3] = \begin{vmatrix} -k_5 & -k_5 & 0 & 0 & 0 & \cdots & 0 \\ k_5 & 0 & -k_5 & 0 & 0 & \cdots & 0 \\ 0 & k_5 & 0 & -k_5 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & k_5 & 0 & -k_5 \\ 0 & 0 & \cdots & 0 & 0 & k_5 & k_5 \end{vmatrix} \tag{76}$$

$$[A_4] = \begin{vmatrix} k_6 & -k_7 & 0 & 0 & 0 & \cdots & 0 \\ -k_7 & k_8 & -k_7 & 0 & 0 & \cdots & 0 \\ 0 & -k_7 & k_8 & -k_7 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & 0 & -k_7 & k_8 & -k_7 \\ 0 & \cdots & 0 & 0 & 0 & -k_7 & k_6 \end{vmatrix} \tag{77}$$

and

$$\left.\begin{aligned}
k_5 &= \frac{\pi}{L} \cdot \frac{G_B b}{t} \cdot \frac{c+t}{2} \\
k_6 &= \frac{\pi^2}{L^2} Ebc + \frac{G_B b}{t} \\
k_7 &= \frac{G_B b}{t} \\
k_8 &= \frac{\pi^2}{L^2} Ebc + \frac{2G_B b}{t}
\end{aligned}\right\} \tag{78}$$

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13. ABSTRACT

The governing equations for laminated beams are developed by variational methods. The beams are considered to consist of n laminae and n-1 bond layers, where n can be any reasonable number greater than one. The bond thicknesses are assumed to be small in comparison with those of the laminae. The effects of shear strains and direct strains normal to the bending axis in the bond layers are included. Two specific problems are solved: bending under uniform load and buckling of simply supported beams. Curves are presented which show the effect of bond flexibility.

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